SIAM J. CONTROL AND OPTIMIZATION
Vol. 28, No. 5, pp. 1190-1208, September 1990

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THE QUADRATIC MATRIX INEQUALITY IN SINGULAR H_∞ CONTROL WITH STATE FEEDBACK*

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Abstract. In this paper the standard H_∞ control problem using state feedback is considered. Given a linear, time-invariant, finite-dimensional system, this problem consists of finding a static state feedback such that the resulting closed-loop transfer matrix has H_∞ norm smaller than some a priori given upper bound. In addition it is required that the closed-loop system is internally stable. Conditions for the existence of a suitable state feedback are formulated in terms of a quadratic matrix inequality, reminiscent of the dissipation inequality of singular linear quadratic optimal control. Where the direct feedthrough matrix of the control input is injective, the results presented here specialize to known results in terms of solvability of a certain indefinite algebraic Riccati equation.

Key words. H_∞ control, state feedback, quadratic matrix inequality, strong controllability, almost disturbance decoupling

AMS(MOS) subject classifications. 93C05, 93C35, 93C45, 93C60, 93B27, 49B99

1. Introduction. In a series of recent papers [1], [2], [5], [8], [10], [15], [18], [23] the by now well-known H_∞ optimal control problem was studied in a perspective of classical linear quadratic optimal control theory. In these papers it is shown that the existence of feedback controllers that result in a closed-loop transfer matrix with H_∞ norm less than a given upper bound is equivalent to the existence of solutions of certain algebraic Riccati equations. Typically, these algebraic Riccati equations are of the type we encounter in the context of linear quadratic differential games.

The first contributions to this new approach in H_∞ optimal control theory were reported in [8], [10], and [23]. These papers deal with the special case where the controllers to be designed are restricted to being state feedback control laws. In later contributions [2], [5], [18] these results were extended to the more general case of dynamic measurement feedback.

If we take a close look at the *type* of conditions for the existence of suitable controllers that are derived in the above references, we see there is a fundamental distinction between two cases. This distinction is tied up with the question of whether or not the *direct feedthrough matrix* of the control input is *injective.* In [10] and [23], *no* assumptions are imposed on the direct feedthrough matrix. The conditions for the existence of a suitable state feedback control law are formulated in terms of a *family* of algebraic Riccati equations, parameterized by a positive real parameter ε. It is shown that there exists an internally stabilizing state feedback control law such that the closed-loop transfer matrix has H_∞ norm less than an a priori given upper bound if and only if there exists a parameter value ε for which the corresponding Riccati equation has a certain solution. In our opinion, a more satisfactory type of condition is obtained in [2], [5], and [18]. In these papers it is assumed that the direct feedthrough matrix of the control input is *injective.* It is then shown that a suitable state feedback control law exists if and only if *one particular* algebraic Riccati equation has a solution with certain properties.

The purpose of the present paper is to reexamine the H_∞ problem with state feedback as studied in [2] and [18], *without making the assumption that the above-mentioned direct feedthrough matrix is injective.* Our aim is to find conditions for the

* Received by the editors March 6, 1989; accepted for publication (in revised form) November 7, 1989.
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existence of suitable state feedback control laws that are of a different type from the one derived in [8], [10], and [23]. Instead our conditions will be of the type proposed in [2] and [18]. Stated differently: we will show how it is possible "to get rid of the parameter ε" in the conditions for the existence of suitable state feedback control laws. Rather than in terms of a particular algebraic Riccati equation, our conditions will be in terms of a certain "quadratic matrix inequality," reminiscent of the dissipation inequality appearing in singular linear quadratic optimal control [4], [13], [19]. It will turn out that the results from [2] and [18] on the special case that the direct feedthrough matrix is injective can be re-obtained from our results.

The outline of this paper is as follows. In § 2 we introduce the problem to be studied and give a statement of our main result. In § 3 we recall some important notions that will be used in this paper. In § 4 we give a description of a decomposition of the input space, the state space and the output space. This decomposition will be instrumental in the proof of our main result. Sections 5 and 6 are devoted to a proof of our main result. Finally, the paper closes with a brief discussion on our results in § 7.

2. Problem formulation and main results. We consider the finite-dimensional, linear, time-invariant system

$$\dot{x} = Ax + Bu + Ew, \qquad z = Cx + Du, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is an unknown disturbance, and $z \in \mathbb{R}^p$ is the output to be controlled. A, B, C, D, and E are real matrices of appropriate dimensions. In this paper we are primarily interested in state feedback. If F is a real $m \times n$ matrix, then the closed-loop transfer matrix resulting from the state feedback control law $u = Fx$ is equal to

$$G_F(s) = (C + DF)(Is - A - BF)^{-1}E.$$

The influence of the disturbance w on the output z is measured by the H_∞ norm of this transfer matrix:

$$\|G_F\|_\infty := \sup_{\omega \in \mathbb{R}} \rho[G_F(i\omega)].$$

Here, $\rho[M]$ denotes the largest singular value of the complex matrix M. The problem that we will study in this paper is the following: given a positive real number γ, find $F \in \mathbb{R}^{m \times n}$ such that

$$\|G_F\|_\infty < \gamma \quad \text{and} \quad \sigma(A + BF) \subset \mathbb{C}^-.$$

Here, $\sigma(M)$ denotes the set of eigenvalues of the matrix M and

$$\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}.$$

A central role in our study of the above problem is played by what we will call the *quadratic matrix inequality.* For any real number $\gamma > 0$ and matrix $P \in \mathbb{R}^{n \times n}$ we define a matrix $F_\gamma(P) \in \mathbb{R}^{(n+m)\times(n+m)}$ by

$$F_\gamma(P) := \begin{pmatrix} PA + A^TP + \gamma^{-2}PEE^TP + C^TC & PB + C^TD \\ B^TP + D^TC & D^TD \end{pmatrix}. \tag{2.2}$$

Clearly, if P is symmetric, then $F_\gamma(P)$ is symmetric as well. If $F_\gamma(P) \geqq 0$, then we will say that *P is a solution to the quadratic matrix inequality at γ.*

In addition to (2.2), for any $\gamma > 0$ and $P \in \mathbb{R}^{n \times n}$ we define a $n \times (n + m)$ polynomial matrix $L_\gamma(P, s)$ by

$$L_\gamma(P, s) := (sI_n - A - \gamma^{-2}EE^TP \quad -B). \tag{2.3}$$

We note that $L_\gamma(P, s)$ is the controllability pencil associated with the system

$$\dot{x} = (A + \gamma^{-2} E E^T P)x + Bu.$$

The transfer matrix of the system Σ given by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{2.4}$$

is equal to the real rational $p \times m$ matrix $G(s) = C(Is - A)^{-1}B + D$. The *normal rank* of a real rational matrix is defined as its rank as a matrix with entries in the field of real rational functions. The normal rank of the transfer matrix G is denoted by normrank G.

In the formulation of our main result we need the concept of *invariant zero* of the system $\Sigma = (A, B, C, D)$. For this definition we refer to § 3 (see also [11]). Finally, let $\mathbb{C}^0 := \{s \in \mathbb{C} \mid \operatorname{Re} s = 0\}$ and let $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. The following is the main result of this paper.

THEOREM 2.1. *Consider the system* (2.1). *Assume that* (A, B, C, D) *has no invariant zeros in* $\mathbb{C}^0$. *Let* $\gamma > 0$. *Then the following two statements are equivalent*:

(i) *There exists* $F \in \mathbb{R}^{m \times n}$ *such that* $\|G_F\|_\infty < \gamma$ *and* $\sigma(A + BF) \subset \mathbb{C}^-$.

(ii) *There exists a real symmetric solution* $P \geqq 0$ *to the quadratic matrix inequality at* γ *such that*

$$\operatorname{rank} F_\gamma(P) = \operatorname{normrank} G \tag{2.5}$$

and

$$\operatorname{rank}\begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + \operatorname{normrank} G \text{ for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+. \tag{2.6}$$

In other words, the existence of a suitable state feedback control law is equivalent to the existence of a particular positive semidefinite solution of the quadratic matrix inequality at γ. This solution should be such that two rank conditions are satisfied.

Before embarking on a proof of this theorem we would like to point out how the results from [2] and [18] for the special case that D is injective can be obtained from our theorem as a special case. First note that in this case we have

$$\operatorname{normrank} G = m.$$

Define

$$R_\gamma(P) := PA + A^T P + \gamma^{-2} P E E^T P + C C^T - (PB + C^T D)(D^T D)^{-1}(B^T P + D^T C).$$

Furthermore, define a real $(n+m) \times (n+m)$ matrix by

$$S(P) := \begin{pmatrix} I_n & -(PB + C^T D)(D^T D)^{-1} \\ 0 & I_m \end{pmatrix}.$$

Then clearly we have

$$S(P) F_\gamma(P) S(P)^T = \begin{pmatrix} R_\gamma(P) & 0 \\ 0 & D^T D \end{pmatrix}.$$

From this we can see that the pair of conditions $F_\gamma(P) \geqq 0$, rank $F_\gamma(P) = m$ is equivalent to the single condition $R_\gamma(P) = 0$. We now analyze the second rank condition appearing in our theorem. It is easily verified that for all $s \in \mathbb{C}$ we have

$$\begin{pmatrix} I_n & 0 & B(D^TD)^{-1} \\ 0 & I_n & -(PB+C^TD)(D^TD)^{-1} \\ 0 & 0 & I_m \end{pmatrix} \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix}$$
$$= \begin{pmatrix} sI - A - \gamma^{-2}EE^TP + B(D^TD)^{-1}(B^TP + D^TC) & 0 \\ R_\gamma(P) & 0 \\ B^TP + D^TC & D^TD \end{pmatrix}.$$

Consequently, if $R_\gamma(P) = 0$ then the condition

$$\operatorname{rank} \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + m \quad \text{for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

is equivalent to

$$\operatorname{rank}\,(sI - A - \gamma^{-2}EE^TP + B(D^TD)^{-1}(B^TP + D^TC)) = n \quad \text{for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

or, equivalently,

$$\sigma(A + \gamma^{-2}EE^TP - B(D^TD)^{-1}(B^TP + D^TC)) \subset \mathbb{C}^-.$$

Thus, for the special case that the direct feedthrough matrix D is injective our main result specializes to Corollary 2.2.

COROLLARY 2.2. *Consider the system* (2.1) *with D injective. Assume that (A, B, C, D) has no invariant zeros in $\mathbb{C}^0$. Let $\gamma > 0$. Then the following two statements are equivalent*:

(i) *There exists $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \gamma$ and $\sigma(A + BF) \subset \mathbb{C}^-$.*

(ii) *There exists a real symmetric solution $P \geqq 0$ to the algebraic Riccati equation*

$$PA + A^TP + \gamma^{-2}PEE^TP + CC^T - (PB + C^TD)(D^TD)^{-1}(B^TP + D^TC) = 0$$

such that

$$\sigma(A + \gamma^{-2}EE^TP - B(D^TD)^{-1}(B^TP + D^TC)) \subset \mathbb{C}^-.$$

A similar result was obtained in [2] and [18] for the special case that $D^TC = 0$ and $D^TD = I_m$. Our result differs slightly from those in [2] and [18] in the sense that we only require P to be semidefinite instead of definite.

3. Preliminaries and notation. In this section we recall some important notions that will be used in the sequel. First, we recall some facts about polynomial matrices. Let $\mathbb{R}[s]$ denote the ring of polynomials with real coefficients. Let $\mathbb{R}^{n \times m}[s]$ be the set of all $n \times m$ matrices with coefficients in $\mathbb{R}[s]$. An element of $\mathbb{R}^{n \times m}[s]$ is called a polynomial matrix. A square polynomial matrix is called unimodular if it is invertible. Two polynomial matrices P and Q are called unimodularly equivalent if there exist unimodular matrices U and V such that $Q = UPV$. In this paper, if P and Q are unimodularly equivalent, then we denote $P \sim Q$. It is well known [3] that for any

$P \in \mathbb{R}^{n \times m}[s]$ there exists $\Psi \in \mathbb{R}^{n \times m}[s]$ of the form

$$\Psi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ \vdots & & & \psi_r & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with ψ_i monic polynomials with the property that ψ_i divides ψ_{i+1}, such that $P \sim \Psi$. The polynomial matrix Ψ is called the *Smith form* of P (see [3]). The polynomials ψ_i are called the invariant factors of P. Their product $\psi := \psi_1 \psi_2 \cdots \psi_r$ is called the zero polynomial of P. The roots of ψ are called the zeros of P. The integer r is equal to the normal rank of P; i.e., $r = \text{normrank } P$. If s is a complex number then $P(s)$ is an element of $\mathbb{C}^{n \times m}$. Its rank is denoted by rank $P(s)$. It is easy to see that normrank $P = \text{rank } P(s)$ for all $s \in \mathbb{C}$ if and only if P is unimodularly equivalent to the constant $n \times m$ matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix.

Next, we recall some important facts on the structure of the linear system Σ given by the equations (2.4). As before, this system is denoted by (A, B, C, D) or simply by Σ. The system matrix of Σ is defined as the polynomial matrix

$$P_\Sigma(s) = \begin{pmatrix} Is - A & -B \\ C & D \end{pmatrix}.$$

The invariant factors of P_Σ are called the *transmission polynomials* of Σ. The zeros of P_Σ are called the *invariant zeros of* Σ. Clearly, $s \in \mathbb{C}$ is an invariant zero of Σ if and only if rank $P_\Sigma(s) <$ normrank P_Σ. It is easy to see that if $F \in \mathbb{R}^{m \times n}$ and if we define $\Sigma_F := (A + BF, B, C + DF, D)$, then $P_\Sigma \sim P_{\Sigma_F}$. In particular this implies that the transmission polynomials of Σ and Σ_F coincide and a fortiori that the invariant zeros of Σ and Σ_F coincide. An important role in this paper is played by *the strongly controllable subspace of* Σ. Consider the following sequence of subspaces:

$$\text{(3.1)} \qquad \begin{aligned} &\mathcal{T}_0(\Sigma) = 0, \\ &\mathcal{T}_{i+1}(\Sigma) = \{x \in \mathbb{R}^n \mid \exists w \in \mathcal{T}_i(\Sigma), u \in \mathbb{R}^m \text{ s.t. } Aw + Bu = x \text{ and } Cw + Du = 0\}. \end{aligned}$$

It is well known (see [7]) that $\mathcal{T}_i(\Sigma)$ $(i = 1, 2, \cdots)$ is a nondecreasing sequence that attains its limit in finitely many steps. The limiting subspace is denoted by $\mathcal{T}(\Sigma)$ and is called the *strongly controllable subspace of* Σ. $\mathcal{T}(\Sigma)$ is known to be the smallest subspace $\mathcal{V}$ of $\mathbb{R}^n$ with the property that there exists a linear mapping G from $\mathbb{R}^p$ to $\mathbb{R}^n$ such that $(A + GC)\mathcal{V} \subseteq \mathcal{V}$ and im $(B + GD) \subseteq \mathcal{V}$. From this it is easily seen that $\mathcal{T}(\Sigma)$ is $(C + DF, A + BF)$-invariant for every linear mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ (a subspace $\mathcal{V}$ is called (C, A)-invariant if it satisfies $A(\mathcal{V} \cap \ker C) \subseteq \mathcal{V}$; see also [12]). The system Σ is called *strongly controllable if* $\mathcal{T}(\Sigma) = \mathbb{R}^n$. If Σ is strongly controllable, then (A, B) is controllable. It is known that Σ is strongly controllable if and only if rank $P_\Sigma(s) = n + \text{rank}\,(C \;\; D)$ for every $s \in \mathbb{C}$ (see [6], [14]). Hence, by the above we find that if $(C \;\; D)$ is surjective, then Σ is strongly controllable if and only if P_Σ is unimodularly equivalent to the constant matrix $(I_{n+p} \;\; 0)$, where I_{n+p} denotes the $(n+p) \times (n+p)$ identity matrix.

We conclude this section by introducing some notation. We will denote $\mathbb{R}^+ := [0, \infty)$. $\mathcal{L}_2(\mathbb{R}^+)$ denotes the space of real-valued measurable functions from $\mathbb{R}^+$ to $\mathbb{R}$ such that $\int_{\mathbb{R}^+} \|x\|^2 \, dt < \infty$. For a given positive integer r we denote by $\mathcal{L}_2^r(\mathbb{R}^+)$ the space of r-vectors with components in $\mathcal{L}_2(\mathbb{R}^+)$. The notation $\| \ \|$ is used for the Euclidean norm on $\mathbb{R}^r$; $\| \ \|_2$ denotes the usual norm on $\mathcal{L}_2^r(\mathbb{R}^+)$; i.e., $\|x\|_2 := (\int_{\mathbb{R}^+} \|x\|^2 \, dt)^{1/2}$.

4. A preliminary feedback transformation. In this section we show that by applying a suitable state feedback transformation $u = F_0 x + v$ to the system $\Sigma = (A, B, C, D)$, it is transformed into a system $\Sigma_{F_0} := (A + BF_0, B, C + DF_0, D)$ with a very particular structure. We will display this structure by writing down the matrices of the mappings $A + BF_0$, B, $C + DF_0$, and D with respect to suitable bases in the input space $\mathbb{R}^m$, the state space $\mathbb{R}^n$, and the output space $\mathbb{R}^p$.

First choose a basis of $\mathbb{R}^m$ as follows. Let $q_1, \cdots, q_l, q_{l+1}, \cdots, q_m$ be a basis such that $q_{l+1}, \cdots, q_m$ is a basis of ker D $(0 \leqq 1 \leqq m)$. In other words, decompose $\mathbb{R}^m = \mathcal{U}_1 \oplus \mathcal{U}_2$, with $\mathcal{U}_2 = \ker D$ and $\mathcal{U}_1$ arbitrary. Next, choose a basis of $\mathbb{R}^p$ as follows. Let $z_1, \cdots, z_r, z_{r+1}, \cdots, z_p$ be an *orthonormal* basis such that $z_1, \cdots, z_r$ is an orthonormal basis of im D and $z_{r+1}, \cdots, z_p$ is an orthonormal basis of $(\operatorname{im} D)^\perp$ $(0 \leqq r \leqq p)$. In other words, write $\mathbb{R}^p = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ with $\mathcal{Z}_1 = \operatorname{im} D$ and $\mathcal{Z}_2 = (\operatorname{im} D)^\perp$. If $\binom{z_1}{z_2}$ is the coordinate vector of a given $z \in \mathbb{R}^p$, then because of orthonormality we have $\|z\| = \|\binom{z_1}{z_2}\|$ (here $\| \ \|$ denotes the Euclidean norm). With respect to these decompositions the mapping D has the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with D_1 invertible. Moreover, B and C can be partitioned as

$$B = (B_1 \quad B_2), \qquad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

It is easy to see that im $B_2 = B \ker D$ and $\ker C_2 = C^{-1} \operatorname{im} D := \{x \mid Cx \in \operatorname{im} D\}$.

Next, define a linear mapping $F_0: \mathbb{R}^n \to \mathbb{R}^m$ by

$$F_0 := \begin{pmatrix} -D_1^{-1} C_1 \\ 0 \end{pmatrix}. \tag{4.1}$$

Then we have

$$C + DF_0 = \begin{pmatrix} 0 \\ C_2 \end{pmatrix}.$$

We now choose a basis of $\mathbb{R}^n$. Let $x_1, \cdots, x_s, x_{s+1}, \cdots, x_t, x_{t+1}, \cdots, x_n$ $(0 \leqq s \leqq t \leqq n)$ be a basis such that $x_{s+1}, \cdots, x_t$ is a basis of $\mathcal{T}(\Sigma) \cap C^{-1} \operatorname{im} D$ and $x_{s+1}, \cdots, x_n$ is a basis of $\mathcal{T}(\Sigma)$. In other words, write $\mathbb{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ with $\mathcal{X}_2 = \mathcal{T}(\Sigma) \cap C^{-1} \operatorname{im} D$, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma)$ and $\mathcal{X}_1$ arbitrary. It turns out that with respect to the bases introduced above, $A + BF_0$, B and $C + DF_0$ have a particular form. This is a consequence of the following lemma.

LEMMA 4.1. *Let F_0 be given by* (4.1). *Then we have*:

(i) $(A + BF_0)(\mathcal{T}(\Sigma) \cap C^{-1} \operatorname{im} D) \subseteq \mathcal{T}(\Sigma)$,

(ii) $\operatorname{im} B_2 \subseteq \mathcal{T}(\Sigma)$,

(iii) $\mathcal{T}(\Sigma) \cap C^{-1} \operatorname{im} D \subseteq \ker C_2$.

Proof. (i) $\mathcal{T}(\Sigma)$ is $(C + DF_0, A + BF_0)$-invariant. This implies that

$$(A + BF_0)(\mathcal{T}(\Sigma) \cap \ker (C + DF_0)) \subseteq \mathcal{T}(\Sigma).$$

Since $\ker(C+DF_0)=\ker C_2=C^{-1}\operatorname{im} D$, the result follows.

(ii) Let $\mathcal{T}_i(\Sigma)$ be the sequence defined by (3.1). Then $\mathcal{T}_1(\Sigma)=B\ker D=\operatorname{im} B_2$. Since $\mathcal{T}_i(\Sigma)$ is nondecreasing this proves our claim.

(iii) This follows immediately from the fact that $C^{-1}\operatorname{im} D=\ker C_2$.

By applying this lemma we find that the matrices of $A+BF_0$, B, $C+DF_0$, and D with respect to the given bases have the following form:

$$
\begin{aligned}
&A+BF_0=\begin{pmatrix} A_{11} & 0 & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}, \qquad B=\begin{pmatrix} B_{11} & 0\\ B_{21} & B_{22}\\ B_{31} & B_{32}\end{pmatrix},\\
&C+DF_0=\begin{pmatrix} 0 & 0 & 0\\ C_{21} & 0 & C_{23}\end{pmatrix}, \qquad D=\begin{pmatrix} D_1 & 0\\ 0 & 0\end{pmatrix}.
\end{aligned}\tag{4.2}
$$

If we apply the feedback transformation $u=F_0x+v$ to the system $\Sigma=(A,B,C,D)$, then the resulting system Σ_{F_0} is given by

$$\dot{x}=(A+BF_0)x+Bv, \qquad z=(C+DF_0)x+Dv. \tag{4.3}$$

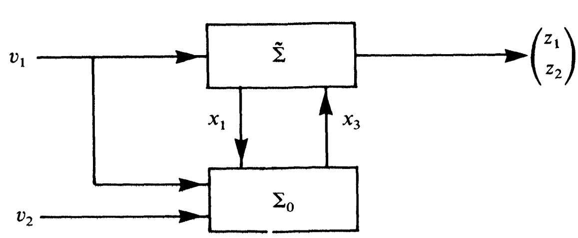
With respect to the given decomposition, let $\binom{v_1}{v_2}$ be the coordinate vector of a given $v\in\mathbb{R}^m$. Likewise, we use the notation $(x_1^T,x_2^T,x_3^T)^T$ and $\binom{z_1}{z_2}$. Then the equations of the system Σ_{F_0} can be arranged in such a way that they have the following form:

$$\dot{x}=A_{11}x_1+(B_{11}\quad A_{13})\begin{pmatrix} v_1\\ x_3\end{pmatrix}, \tag{4.4}$$

$$\begin{pmatrix}\dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix} A_{22} & A_{23}\\ A_{32} & A_{33}\end{pmatrix}\begin{pmatrix} x_2\\ x_3\end{pmatrix}+\begin{pmatrix} B_{22}\\ B_{32}\end{pmatrix}v_2+\begin{pmatrix} B_{21} & A_{21}\\ B_{31} & A_{31}\end{pmatrix}\begin{pmatrix} v_1\\ x_1\end{pmatrix}, \tag{4.5}$$

$$\begin{pmatrix} z_1\\ z_2\end{pmatrix}=\begin{pmatrix} 0\\ C_{21}\end{pmatrix}x_1+\begin{pmatrix} D_1 & 0\\ 0 & C_{23}\end{pmatrix}\begin{pmatrix} v_1\\ x_3\end{pmatrix}. \tag{4.6}$$

As already suggested by the way that we have arranged these equations, the system Σ_{F_0} can be considered as the interconnection of two subsystems. This is depicted as follows:



Here,

$$\tilde{\Sigma}:=\left(A_{11},(B_{11}\quad A_{13}),\ \begin{pmatrix} 0\\ C_{21}\end{pmatrix},\begin{pmatrix} D_1 & 0\\ 0 & C_{23}\end{pmatrix}\right) \tag{4.7}$$

is the system given by (4.4) and (4.6). It has input space $\mathcal{U}_1\times\mathcal{X}_3$, state space $\mathcal{X}_1$, and output space $\mathbb{R}^p$. Σ_0 is the system given by (4.5). It has input space $\mathbb{R}^m\times\mathcal{X}_1$ and state space $\mathcal{X}_2\oplus\mathcal{X}_3$. The interconnection is made via x_1 and x_3, as depicted above. Note that $\tilde{\Sigma}$ and Σ_{F_0} have the same output equation. However, in Σ_{F_0} the variable x_3 is generated by Σ_0, whereas in $\tilde{\Sigma}$ it is considered as an input and is free. The systems Σ_0 and $\tilde{\Sigma}$ turn out to have a couple of nice structural properties, as shown in Lemma 4.2.

LEMMA 4.2. (i) *C_{23} is injective,*

(ii) *The system*

$$\Sigma_1 := \left(\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, (0 \quad I), 0 \right) \tag{4.8}$$

with input space $\mathscr{U}_2$, state space $\mathscr{X}_2 \oplus \mathscr{X}_3$ $(=\mathscr{T}(\Sigma))$, and output space $\mathscr{X}_3$ is strongly controllable.

Proof. (i) Let $(x_1^T, x_2^T, x_3^T)^T$ be the coordinate vector of a given $x \in \mathbb{R}^n$. Assume that $C_{23}x_3 = 0$. Let $\tilde{x} \in \mathbb{R}^n$ be the vector with coordinates $(0^T, 0^T, x_3^T)^T$. Then $\tilde{x} \in \mathscr{X}_3$. In addition, $\tilde{x} \in \mathscr{T}(\Sigma) \cap \ker C_2 = \mathscr{X}_2$. Thus $\tilde{x} = 0$, so $x_3 = 0$.

(ii) Let $\mathscr{T}(\Sigma_1)$ be the strongly controllable subspace of the system Σ_1 given by (4.8). We will prove that $\mathscr{T}(\Sigma_1) = \mathscr{X}_2 \oplus \mathscr{X}_3$. First note that there exists $G = \binom{G_2}{G_3}$ such that

$$\left(\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} + \begin{pmatrix} G_2 \\ G_3 \end{pmatrix} (0 \quad I) \right) \mathscr{T}(\Sigma_1) \subseteq \mathscr{T}(\Sigma_1).$$

Also note that

$$\operatorname{im} \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} \subseteq \mathscr{T}(\Sigma_1).$$

Now assume that $\mathscr{T}(\Sigma_1) \subseteq \mathscr{X}_2 \oplus \mathscr{X}_3$ with *strict* inclusion. Define $\mathscr{V} \subseteq \mathbb{R}^n$ by

$$\mathscr{V} := \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in \mathscr{T}(\Sigma_1) \right\}.$$

Clearly, $\mathscr{V} \subseteq \mathscr{T}(\Sigma)$ with strict inclusion. We claim that there exists a linear map $G_0 : \mathbb{R}^p \to \mathbb{R}^n$ such that

$$(A + G_0 C)\mathscr{V} \subseteq \mathscr{V} \tag{4.9}$$

$$\operatorname{im}(B + G_0 D) \subseteq \mathscr{V}. \tag{4.10}$$

Indeed, let C_{23}^+ be any left inverse of C_{23} and define

$$G_0 := \begin{pmatrix} B_{11} & -A_{13} \\ B_{21} & G_2 \\ B_{31} & G_3 \end{pmatrix} \begin{pmatrix} -D_1^{-1} & 0 \\ 0 & C_{23}^+ \end{pmatrix}.$$

It is then straightforward to verify (4.9) and (4.10). This, however, contradicts the fact that $\mathscr{T}(\Sigma)$ is the smallest subspace $\mathscr{V}$ for which (4.9) and (4.10) hold (see § 3). We conclude that $\mathscr{X}_2 \oplus \mathscr{X}_3 = \mathscr{T}(\Sigma_1)$. □

Our next result states that the zero structure of the original system $\Sigma = (A, B, C, D)$ is completely determined by the zero structure of the subsystem $\tilde{\Sigma}$ given by (4.7). A transmission polynomial of a system is called *nontrivial* if it is unequal to the constant polynomial 1.

LEMMA 4.3. *The nontrivial transmission polynomials of Σ and $\tilde{\Sigma}$, respectively, coincide.*

Proof. According to § 3 the transmission polynomials of Σ and Σ_{F_0} coincide. Thus, to prove the lemma it suffices to show that the system matrix P_0 of Σ_{F_0} is unimodularly equivalent to a polynomial matrix of the form

$$\begin{pmatrix} P_{\tilde{\Sigma}}(s) & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

where $P_{\tilde{\Sigma}}(s)$ is the system matrix of $\tilde{\Sigma}$. Since Σ_1 is strongly controllable and $(0 \;\; I)$ is surjective, the Smith form of P_{Σ_1} is equal to $(I_1 \;\; 0)$ (I_1 denotes the identity matrix with size equal to $\dim \mathscr{X}_2 + 2\dim \mathscr{X}_3$). In addition, clearly we have

$$P_{\Sigma_1} \sim \begin{pmatrix} sI - A_{22} & 0 & -B_{22} \\ -A_{32} & 0 & -B_{32} \\ 0 & I & 0 \end{pmatrix} \sim \begin{pmatrix} sI - A_{22} & -B_{22} & 0 \\ -A_{32} & -B_{32} & 0 \\ 0 & 0 & I \end{pmatrix},$$

so we conclude that

$$\begin{pmatrix} sI - A_{22} & -B_{22} \\ -A_{32} & -B_{32} \end{pmatrix}$$

is unimodularly equivalent to $(I_2 \;\; 0)$. Here I_2 denotes the identity matrix of size $\dim \mathscr{X}_2 + \dim \mathscr{X}_3$. The proof is then completed by noting that

$$P_0 \sim \begin{pmatrix} sI - A_{11} & -B_{11} & -A_{13} & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ C_{21} & 0 & C_{23} & 0 & 0 \\ -A_{21} & -B_{21} & -A_{23} & sI - A_{22} & -B_{22} \\ -A_{31} & -B_{31} & sI - A_{33} & -A_{32} & -B_{32} \end{pmatrix} \sim \begin{pmatrix} P_{\tilde{\Sigma}}(s) & 0 & 0 \\ 0 & I_2 & 0 \end{pmatrix}.$$

A consequence of the above lemma is that the invariant zeros of Σ and $\tilde{\Sigma}$, respectively, coincide.

Our next lemma states that the normal rank of the transfer matrix $G(s) = C(sI - A)^{-1}B + D$ of the system Σ is equal to the number $\operatorname{rank} D_1 + \dim \mathscr{X}_3$ or, equivalently, Lemma 4.4.

LEMMA 4.4. *We have*

$$\operatorname{normrank} G = \operatorname{rank} \begin{pmatrix} C_{23} & 0 \\ 0 & D_1 \end{pmatrix}.$$

Proof. Define $L(s) := sI - A$. Then we have

$$\operatorname{normrank} \begin{pmatrix} L & 0 \\ 0 & G \end{pmatrix} = n + \operatorname{normrank} G. \tag{4.11}$$

We also have

$$\begin{pmatrix} I & 0 \\ C(sI-A)^{-1} & I \end{pmatrix} \begin{pmatrix} L(s) & 0 \\ 0 & G(s) \end{pmatrix} \begin{pmatrix} I & -(sI-A)^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ F_0 & I \end{pmatrix}$$

$$= \begin{pmatrix} sI - (A + BF_0) & -B \\ C + DF_0 & D \end{pmatrix} = \begin{pmatrix} sI - A_{11} & 0 & -A_{13} & -B_{11} & 0 \\ -A_{21} & sI - A_{22} & -A_{23} & -B_{21} & -B_{22} \\ -A_{31} & -A_{32} & sI - A_{33} & -B_{31} & -B_{32} \\ 0 & 0 & 0 & D_1 & 0 \\ C_{21} & 0 & C_{23} & 0 & 0 \end{pmatrix}.$$

Since C_{23} and D_1 are injective, we can make the $(1,3)$, $(1,4)$, $(2,4)$, and $(3,4)$ blocks zero by unimodular transformations. Furthermore, we can make a basis transformation on the output such that C_{23} has the form $\binom{I_r}{0}$ where $r = \operatorname{rank} C_{23}$. Thus, after suitable permutation of blocks, the normal rank of the latter matrix turns out to be equal to the normal rank of

$$\begin{pmatrix} \boxed{sI-\tilde{A}_{11}} & 0 & 0 & 0 & 0 \\ -A_{21} & sI-A_{22} & -A_{23} & -B_{22} & 0 \\ -A_{31} & -A_{32} & sI-A_{33} & -B_{32} & 0 \\ C_{211} & 0 & I_r & 0 & 0 \\ C_{212} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{D_1} \end{pmatrix}.$$

Here $\tilde{A}_{11}$ is a given matrix. Since, by Lemma 4.2, the matrix in the center has full row rank for all $s \in \mathbb{C}$ and since normrank $(sI-\tilde{A}_{11}) = \dim \mathscr{X}_1$, we find

$$\operatorname{normrank}\begin{pmatrix} L & 0 \\ 0 & G \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} C_{23} & 0 \\ 0 & D_1 \end{pmatrix}.$$

Combining this with (4.11), we obtain the desired result. □

To conclude this section we want to note that if D is injective, then the subspace $\mathscr{U}_2$ in the decomposition of $\mathbb{R}^m$ vanishes. Consequently, the partitioning of B reduces to a single block and the partitioning of D reduces to $\binom{D_1}{0}$ with D_1 invertible. It is left as an exercise to the reader to show that $\mathscr{T}(\Sigma) = 0$ if and only if $\ker D \subseteq \ker B$. Thus, if D is injective, then also $\mathscr{T}(\Sigma) = 0$. In that case the subspaces $\mathscr{X}_2$ and $\mathscr{X}_3$ appearing in the decomposition of $\mathscr{X}$ both vanish and the partitioning of $A + BF_0$ reduces to a single block.

5. Solvability of the quadratic matrix inequality. In this section we will establish a proof of the implication (i)$\Rightarrow$(ii) in Theorem 2.1: assuming that a suitable state feedback control law exists, we show that the quadratic matrix inequality has a solution with the asserted properties.

Consider our control system (2.1). For given disturbance and control functions w and u we denote by $x_{w,u}$ and $z_{w,u}$ the corresponding state trajectory and output function, respectively, with $x(0) = 0$. We will first formulate a theorem that serves as a basis for the developments in the rest of this paper. The theorem is concerned with the special case that in the system (2.1) the direct feedthrough matrix D is injective. The result in Theorem 5.1 is a generalization of [2, Thm. 2] and of results in [18].

THEOREM 5.1. *Consider the system* (2.1) *and assume that D is injective. Assume that (A, B, C, D) has no invariant zeros in $\mathbb{C}^0$. Let $\gamma > 0$. Then the following statements are equivalent*:

(i) *(A, B) is stabilizable and, in addition, there exists $\delta > 0$ such that for all $w \in \mathscr{L}_2^l(\mathbb{R}^+)$ there exists $u \in \mathscr{L}_2^m(\mathbb{R}^+)$ for which $x_{w,u} \in \mathscr{L}_2^n(\mathbb{R}^+)$ and $\|z_{w,u}\|_2 \leqq (\gamma - \delta)\|w\|_2$.*

(ii) *There exists a real symmetric solution $P \geqq 0$ to the algebraic Riccati equation*

$$PA + A^TP + \gamma^{-2}PEE^TP + C^TC - (PB + C^TD)(D^TD)^{-1}(B^TP + D^TC) = 0 \tag{5.1}$$

such that

$$\sigma(A + \gamma^{-2}EE^TP - B(D^TD)^{-1}(B^TP + D^TC)) \subset \mathbb{C}^-. \tag{5.2}$$

Moreover, if the latter holds, then one possible choice for u is given by $u = Fx$, with

$$F = -(D^T D)^{-1}(B^T P + D^T C).$$

For this F we have $\|G_F\|_\infty < \gamma$ and $\sigma(A+BF) \subset \mathbb{C}^-$.

Proof. A proof of this theorem can be based on the proof of [18, Thm. 2.1c]. In the latter paper it is assumed that C is injective and that $C^T D = 0$, which implies that (A, B, C, D) has no zeros at all. The proof of Theorem 2.1c of [18] can, however, be modified to yield a proof of our result. In doing this the following important point might need clarification. Since, in our context (C, A) is not necessarily detectable, we must make a careful distinction between the H_∞ problem with stability (i.e., $x \in \mathscr{L}_2^n$ and $u \in \mathscr{L}_2^m$) and the H_∞ problem without stability (i.e., no restrictions on x and u). In the proof of Theorem 2.1 of [18] a version of the maximum principle is used that gives a *sufficient* condition for optimality in the case that (C, A) is detectable (for a finite-horizon version of this result see [9, Chap. 5.2]). However, if we drop the detectability assumption, this method can still be used for the H_∞ problem with stability. The remainder of the proof in [18] can be checked step by step and remains valid.

Since in our context (C, A) is not necessarily observable (in contrast with [2] and [18]) our theorem involves a semidefinite solution of (5.1) rather than a definite one. □

Now, again consider the system (2.1), this time without making any assumptions on the matrix D. Choose bases in the state space, the input space, and the output space as in § 4 and apply the feedback transformation $u = F_0 x + v$, with F_0 given by (4.1). After this transformation we have

$$\dot{x} = (A + BF_0)x + Bv + Ew, \qquad z = (C + DF_0)x + Dv. \tag{5.3}$$

If we partition $E = (E_1^T, E_2^T, E_3^T)^T$, then in terms of our decomposition (5.3) can be written as follows:

$$\dot{x}_1 = A_{11}x_1 + (B_{11} \quad A_{13})\begin{pmatrix} v_1 \\ x_3 \end{pmatrix} + E_1 w, \tag{5.4}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} B_{21} & A_{21} \\ B_{31} & A_{31} \end{pmatrix}\begin{pmatrix} v_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} w, \tag{5.5}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ C_{21} \end{pmatrix} x_1 + \begin{pmatrix} D_1 & 0 \\ 0 & C_{23} \end{pmatrix}\begin{pmatrix} v_1 \\ x_3 \end{pmatrix}. \tag{5.6}$$

For given disturbance and control functions w and v, let $x_{w,v}$ and $z_{w,v}$ denote the state trajectory and output, respectively, of (5.3), with $x(0) = 0$. The idea that we want to pursue is the following. If there exists a feedback law $u = Fx$ for (2.1) such that $\|G_F\|_\infty < \gamma$ and $\sigma(A+BF) \subset \mathbb{C}^-$, then the feedback law $v = (F - F_0)x$ in (5.3) yields a closed-loop transfer matrix from w to z with H_∞ norm smaller than γ. In other words,

$$\beta := \sup_{w \in \mathscr{L}_2^l(\mathbb{R}^+)} \frac{\|z_{w,v}\|_2}{\|w\|_2} < \gamma. \tag{5.7}$$

Also, $x_{w,v} \in \mathscr{L}_2^n(\mathbb{R}^+)$. Let $\delta := \gamma - \beta$. Then, for a given w, define v_1 as the first component of $v = (F - F_0)x_{w,v}$ and take x_3 as the third component of $x_{w,v}$. Interpret $\binom{v_1}{x_3}$ as an input for the subsystem $\tilde{\Sigma}$ defined by (5.4) and (5.6). It then follows from (5.7) that

$$\left\| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|_2 \leqq (\gamma - \delta)\|w\|_2.$$

Moreover, the "input" $\binom{v_1}{x_3}$ and the "state trajectory" x_1 are in $\mathscr{L}_2$. The crucial observation is now that the direct feedthrough matrix of $\tilde{\Sigma}$ is *injective* (see Lemma 4.2). Thus we can apply Theorem 5.1 to establish the existence of a solution to the algebraic Riccati equation associated with the system $\tilde{\Sigma}$. Before doing this, however, we should make sure that $(A_{11}, (B_{11}, A_{13}))$ is stabilizable and that $\tilde{\Sigma}$ given by (4.7) has no invariant zeros in $\mathbb{C}^0$. It is easily seen that if (A, B) is stabilizable, then also $(A_{11}, (B_{11}, A_{13}))$ is stabilizable. Furthermore, if $\Sigma = (A, B, C, D)$ has no invariant zeros in $\mathbb{C}^0$, then the same holds for $\tilde{\Sigma}$ (see Lemma 4.3). Consequently, we have the following corollary.

COROLLARY 5.2. *Consider the system* (2.1). *Assume that* (A, B, C, D) *has no invariant zeros in* $\mathbb{C}^0$. *Let* $\gamma > 0$ *and assume there exists* $F \in \mathbb{R}^{m\times n}$ *such that* $\|G_F\|_\infty < \gamma$ *and* $\sigma(A+BF) \subset \mathbb{C}^-$. *Then there exists a real symmetric solution* $P_{11} \geqq 0$ *to the algebraic Riccati equation*

$$(5.8)\quad \begin{aligned} &P_{11}A_{11} + A_{11}^TP_{11} + C_{21}^TC_{21} + \gamma^{-2}P_{11}E_1E_1^TP_{11} - P_{11}B_{11}(D_1^TD_1)^{-1}B_{11}^TP_{11} \\ &\quad -(A_{13}^TP_{11} + C_{23}^TC_{21})^T(C_{23}^TC_{23})^{-1}(A_{13}^TP_{11} + C_{23}^TC_{21}) = 0 \end{aligned}$$

such that

$$(5.9)\quad \begin{aligned} &\sigma(A_{11} + \gamma^{-2}E_1E_1^TP_{11} - B_{11}(D_1^TD_1)^{-1}B_{11}^TP_{11} \\ &\quad - A_{13}(C_{23}^TC_{23})^{-1}(A_{13}^TP_{11} + C_{23}^TC_{21})) \subset \mathbb{C}^-. \end{aligned}$$

Our next step is to establish a connection between the algebraic Riccati equation (5.8) and the quadratic matrix inequality.

It turns out that there is a one-to-one correspondence between the set of solutions to (5.8) and the set of solutions to the quadratic matrix inequality at γ that satisfy the rank condition (2.5). To prove this, we need the following lemma.

LEMMA 5.3. *Assume* $P \in \mathbb{R}^{n\times n}$ *is a solution to the quadratic matrix inequality at* γ. *Then* $\mathscr{T}(\Sigma) \subseteq \ker P$.

Proof. Let F_0 be given by (4.1). Let $\mathscr{R}$ be the smallest $(C+DF_0, A+BF_0)$-invariant subspace containing $B \ker D$. We claim that $\mathscr{R} = \mathscr{T}(\Sigma)$. We know that $\mathscr{T}(\Sigma)$ is $(C+DF, A+BF)$-invariant for all F and hence also for $F = F_0$. Second, by Lemma 4.1(ii) we have $\mathscr{T}(\Sigma) \supseteq \ker D$. Therefore, we have $\mathscr{R} \subseteq \mathscr{T}(\Sigma)$. Conversely, we know that

$$\exists G_1 : \operatorname{im}(C+DF_0) \to \mathbb{R}^n \text{ s.t. } [(A+BF_0) + G_1(C+DF_0)]\mathscr{R} \subseteq \mathscr{R},$$

$$\exists G_2 : \operatorname{im} D \to \mathbb{R}^n \text{ s.t. } \operatorname{im}(B+G_2D) = B\ker D \subseteq \mathscr{R}.$$

Since $D^T(C+DF_0) = 0$ (this can be checked easily) we can find a linear mapping G such that $G|_{\operatorname{im}(C+DF_0)} = G_1$ and $G|_{\operatorname{im} D} = G_2$ and hence we have found a G such that $(A+GC)\mathscr{R} \subseteq \mathscr{R}$ and $\operatorname{im}(B+GD) \subseteq \mathscr{R}$. Thus we find $\mathscr{R} \supseteq \mathscr{T}(\Sigma)$ and hence $\mathscr{R} = \mathscr{T}(\Sigma)$.

Let $\gamma > 0$ and define

$$(5.10)\quad M_\gamma(P) := \begin{pmatrix} I & F_0^T \\ 0 & I \end{pmatrix} F_\gamma(P) \begin{pmatrix} I & 0 \\ F_0 & I \end{pmatrix}.$$

If $F_\gamma(P) \geqq 0$ then also

$$(5.11)\quad M_\gamma(P) = \begin{pmatrix} P(A+BF_0) + (A+BF_0)^TP + \gamma^{-2}PEE^TP + (C+DF_0)^T(C+DF_0) & PB \\ B^TP & D^TD \end{pmatrix} \geqq 0$$

We claim that $B \ker D \subseteq \ker P$. Let $u \in \mathbb{R}^m$ be such that $Du = 0$. Then we find $\binom{0}{u}^T M_\gamma(P)\binom{0}{u} = 0$ and hence, since $M_\gamma(P) \geqq 0$, we find $M_\gamma(P)\binom{0}{u} = 0$. This implies $PBu = 0$. Next we have that $\ker P$ is $(C+DF_0, A+BF_0)$-invariant. Assume that $x \in$

$\ker P \cap \ker(C+DF_0)$. Then

$$x^T(P(A+BF_0)+(A+BF_0)^TP+\gamma^{-2}PEE^TP+(C+DF_0)^T(C+DF_0))x=0.$$

Hence, by applying x to one side only, we find $P(A+BF_0)x=0$ and therefore $(A+BF_0)x \in \ker P$. Since $\mathcal{T}(\Sigma)$ is the smallest space with these two properties, we must have $\mathcal{T}(\Sigma) \subseteq \ker P$. □

Using the above lemma, we now obtain the following result.

THEOREM 5.4. *Let $\gamma > 0$ and $P \in \mathbb{R}^{n\times m}$. The following two statements are equivalent:*

(i) *P is a symmetric solution to the quadratic matrix inequality at γ such that* rank $F_\gamma(P) =$ normrank G.

$$\text{(ii)} \qquad P=\begin{pmatrix} P_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where P_{11} is a symmetric matrix satisfying (5.8).

Furthermore, if the above holds, then the following two statements are equivalent:

$$\text{(iii)} \qquad \operatorname{rank}\begin{pmatrix} L_\gamma(P,s)\\ F_\gamma(P)\end{pmatrix} = n + \operatorname{normrank} G \text{ for all } s \in \mathbb{C}^0 \cup \mathbb{C}^+.$$

$$\text{(iv)} \qquad \sigma(A_{11}+\gamma^{-2}E_1E_1^TP_{11}-B_{11}(D_1^TD_1)^{-1}B_{11}^TP_{11} \\ -A_{13}(C_{23}^TC_{23})^{-1}(A_{13}^TP_{11}+C_{23}^TC_{21}))\subset \mathbb{C}^-.$$

Proof. By (5.10) we have $M_\gamma(P) \geqq 0$ if and only if $F_\gamma(P) \geqq 0$, and we also know that these matrices have the same rank. Assume a symmetric P satisfies $M_\gamma(P) \geq 0$ and rank $M_\gamma(P) =$ normrank G. Since $P\mathcal{T}(\Sigma)=0$ (see Lemma 5.3) we know that we can write P as

$$P=\begin{pmatrix} P_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{5.12}$$

If we also use the decompositions (4.2) for the other matrices, then we find that $M_\gamma(P)$ is equal to

$$\begin{pmatrix} P_{11}A_{11}+A_{11}^TP_{11}+C_{21}^TC_{21}+\gamma^{-2}P_{11}E_1E_1^TP_{11} & 0 & P_{11}A_{13}+C_{21}^TC_{23} & P_{11}B_{11} & 0\\ 0 & 0 & 0 & 0 & 0\\ A_{13}^TP_{11}+C_{23}^TC_{21} & 0 & C_{23}^TC_{23} & 0 & 0\\ B_{11}^TP_{11} & 0 & 0 & D_1^TD_1 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \geqq 0. \tag{5.13}$$

According to Lemma 4.4 the rank of this matrix equals the rank of the encircled matrix. Thus, the Schur complement of the encircled matrix must be equal to zero. Since this condition exactly yields the algebraic Riccati equation (5.8) we find that P_{11} is a solution of (5.8).

Conversely, if P_{11} is a solution of (5.8), then the Schur complement of the encircled matrix in (5.13) is zero. Therefore, it satisfies the matrix inequality (5.13), and the rank of the matrix is equal to normrank G. Hence P given by (5.12) satisfies the required properties.

Now assume that (i) or (ii) holds. We will prove the equivalence of (iii) and (iv). Denote the matrix in (iv) by Z. We will apply the following unimodular transformation to the matrix in (iii):

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & F_0^T \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix}\begin{pmatrix} I & 0 \\ F_0 & I \end{pmatrix}.$$

When we use the decompositions in (4.2), the latter turns out to be equal to

$$\begin{pmatrix} sI-A_{11}-\gamma^{-2}E_1E_1^TP_{11} & 0 & -A_{13} & -B_{11} & 0 \\ -A_{21}-\gamma^{-2}E_2E_1^TP_{11} & sI-A_{22} & -A_{23} & -B_{21} & -B_{22} \\ -A_{31}-\gamma^{-2}E_3E_1^TP_{11} & -A_{32} & sI-A_{33} & -B_{31} & -B_{32} \\ P_{11}A_{11}+A_{11}^TP_{11}+C_{21}^TC_{21}+\gamma^{-2}P_{11}E_1E_1^TP_{11} & 0 & P_{11}A_{13}+C_{21}^TC_{23} & P_{11}B_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{13}^TP_{11}+C_{23}^TC_{21} & 0 & C_{23}^TC_{23} & 0 & 0 \\ B_{11}^TP_{11} & 0 & 0 & D_1^TD_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.14}$$

By using Schur complements, we can get the Riccati equation (5.8) in the 4,1 position and the matrix Z in the 1,1 position of the above matrix. Furthermore, since $D_1^TD_1$ is invertible, we can make the 2,4 and 3,4 blocks equal to zero by a unimodular transformation. Since P_{11} is a solution of the Riccati equation, the 4,1 block becomes zero. Thus, we find that (5.14) is unimodularly equivalent to

$$\begin{pmatrix} sI-Z & 0 & 0 & 0 & 0 \\ * & sI-A_{22} & -A_{23} & 0 & -B_{22} \\ * & -A_{32} & sI-A_{33} & 0 & -B_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{23}^TC_{23} & 0 & 0 \\ 0 & 0 & 0 & D_1^TD_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $*$ denotes matrices that are unimportant for this argument.

Now by Lemma 4.2 the encircled matrices together form the system matrix of a strongly controllable system. Hence this system matrix is unimodularly equivalent to a constant matrix $(I\ 0)$, where I denotes the identity matrix of appropriate size. Therefore, we can make the 2,1 and 3,1 blocks zero by a unimodular transformation. Thus, after reordering we find

$$\begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} \sim \begin{pmatrix} \boxed{sI-Z} & 0 & 0 & 0 & 0 \\ 0 & sI-A_{22} & -A_{23} & -B_{22} & 0 \\ 0 & -A_{32} & sI-A_{33} & -B_{32} & 0 \\ 0 & 0 & C_{23}^TC_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{D_1^TD_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the matrix on the left has rank $n+\text{normrank}\, G$ for all $s\in\mathbb{C}^0\cup\mathbb{C}^+$ if and only if $\sigma(Z)\subset\mathbb{C}^-$. This proves that (iii) and (iv) are equivalent. □

A proof of the implication (i)⇒(ii) in Theorem 2.1 is now obtained immediately by combining Corollary 5.2 and Theorem 5.4.

6. Existence of state feedback control laws. In this section we give a proof of the implication (ii)⇒(i) in Theorem 2.1. We first explain the idea of the proof. Again, we consider our control system (5.3) as the interconnection of the subsystem $\tilde{\Sigma}$ given by (5.4), (5.6) and the subsystem Σ_0 given by (5.5). Suppose that the quadratic matrix inequality has a positive-semidefinite solution at γ such that the rank conditions (2.5) and (2.6) hold. Then according to Theorem 5.4, the algebraic Riccati equation associated with the subsystem $\tilde{\Sigma}$ has a positive-semidefinite solution P_{11} such that (iv) of Theorem 5.4 holds. Thus by applying Theorem 5.1 to the subsystem $\tilde{\Sigma}$, we find that the "feedback law"

$$v_1 = -(D_1^T D_1)^{-1} B_{11}^T P_{11} x_1, \tag{6.1}$$

$$x_3 = -(C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) x_1, \tag{6.2}$$

yields a closed-loop transfer matrix for $\tilde{\Sigma}$ with H_∞ norm smaller than γ. Now we will do the following: construct a state feedback law for the *original* system (5.3) in such a way that in the subsystem $\tilde{\Sigma}$ the equality (6.2) holds *approximately*. The closed-loop transfer matrix of the original system will then be approximately equal to that of the subsystem $\tilde{\Sigma}$ and will therefore also have H_∞ norm smaller than γ.

In our proof an important role will be played by a result in the context of the problem of *almost disturbance decoupling* as studied in [19] and [22]. We will first recall this result here. For the moment assume that we have the following system:

$$\dot{x} = Ax + Bu + Ew, \qquad z = Cx. \tag{6.3}$$

For this system, the almost disturbance decoupling problem with pole placement (ADDPPP) is formulated as follows. For all $\varepsilon > 0$ and for all $M \in \mathbb{R}$, find $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \varepsilon$ and $\sigma(A+BF) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < M\}$. It is shown in [19] and [22] that conditions for the existence of such F can be stated in terms of the strongly controllable subspace $\mathcal{T}(\Sigma)$ associated with the system $\Sigma = (A, B, C, 0)$. (In fact, in [19] and [22] this subspace is denoted by $\mathcal{R}_b^*(\ker C)$.) The exact result is as follows.

LEMMA 6.1. *Consider the system* (6.3). *Let $\mathcal{T}(\Sigma)$ denote the strongly controllable subspace associated with $\Sigma = (A, B, C, 0)$. Then the following two statements are equivalent*:

(i) *For all $\varepsilon > 0$ and for all $M \in \mathbb{R}$ there exists $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \varepsilon$ and $\sigma(A+BF) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < M\}$.*

(ii) im $E \subset \mathcal{T}(\Sigma)$ *and* (A, B) *is controllable.*

As an immediate consequence of the above we obtain the following fact. If $\Sigma = (A, B, C, 0)$ is strongly controllable, then for all $\varepsilon > 0$ and for all $M \in \mathbb{R}$ there exists $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \varepsilon$ and $\sigma(A+BF) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < M\}$. Thus, in particular, if $\Sigma = (A, B, C, 0)$ is strongly controllable, then for all $\varepsilon > 0$ there exists $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \varepsilon$ and $\sigma(A+BF) \subset \mathbb{C}^-$.

We now formulate and prove the converse of Corollary 5.2.

THEOREM 6.2. *Consider the system* (2.1). *Assume that (A, B, C, D) has no invariant zeros in $\mathbb{C}^0$. Let $\gamma > 0$. Assume there exists a real symmetric solution $P_{11} \geqq 0$ to the algebraic Riccati equation* (5.8) *such that* (5.9) *holds. Then there exists $F \in \mathbb{R}^{m \times n}$ such that $\|G_F\|_\infty < \gamma$ and $\sigma(A+BF) \subset \mathbb{C}^-$.*

Proof. Clearly, it is sufficient to prove the existence of such a state feedback law as $v = Fx$ for the system (5.3). Let this system be decomposed according to (5.4)-(5.6). Choose

$$v_1 = -(D_1^T D_1)^{-1} B_{11}^T P_{11} x_1$$

and introduce a new state variable q_3 by

$$q_3 := x_3 + (C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) x_1.$$

Then (5.4)-(5.6) can be rewritten as

$$\dot{x}_1 = \tilde{A}_{11} x_1 + A_{13} q_3 + E_1 w, \tag{6.4}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} E_2 \\ \tilde{E}_3 \end{pmatrix} w, \tag{6.5}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ C_{23} \end{pmatrix} q_3. \tag{6.6}$$

Here we use the following definitions:

$$\begin{aligned}
\tilde{A}_{11} &:= A_{11} - A_{13}(C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) - B_{11}(D_1^T D_1)^{-1} B_{11}^T P_{11}, \\
\tilde{A}_{21} &:= A_{21} - A_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) - B_{21}(D_1^T D_1)^{-1} B_{11}^T P_{11}, \\
\tilde{A}_{31} &:= A_{31} - A_{33}(C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) - B_{31}(D_1^T D_1)^{-1} B_{11}^T P_{11} \\
&\qquad + (C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) \tilde{A}_{11}, \\
\tilde{A}_{33} &:= A_{33} + (C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) A_{13}, \\
\tilde{C}_1 &:= -D_1(D_1^T D_1)^{-1} B_{11}^T P_{11}, \\
\tilde{C}_2 &:= C_{21} - C_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}), \\
\tilde{E}_3 &:= E_3 + (C_{23}^T C_{23})^{-1}(A_{13}^T P_{11} + C_{23}^T C_{21}) E_1.
\end{aligned}$$

According to Theorem 5.1, if in the subsystem formed by (6.4) and (6.6) we have $q_3 = 0$, then its transfer matrix from w to z has H_∞ norm smaller than γ. Moreover, we have $\sigma(\tilde{A}_{11}) \subset \mathbb{C}^-$. Hence, there exist $M > 0$ and $\rho > 0$ such that for all w and q_3 in $\mathcal{L}_2$, we have

$$\|z\|_2 < (\gamma - \rho)\|w\|_2 + M\|q_3\|_2. \tag{6.7}$$

Also by the fact that $\tilde{A}_{11}$ is stable, there exist $M_1, M_2 > 0$ such that for all w and q_3 in $\mathcal{L}_2$, we have

$$\|x_1\|_2 < M_1\|w\|_2 + M_2\|q_3\|_2. \tag{6.8}$$

We claim that the following system is strongly controllable:

$$\left(\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, (0 \quad I), 0 \right). \tag{6.9}$$

This can be seen by the following transformation:

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & A_{33}-\tilde{A}_{33} \\ 0 & 0 & I \end{pmatrix}\left(\begin{array}{cc:c} sI-A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI-A_{33} & -B_{32} \\ \hdashline 0 & I & 0 \end{array}\right)=\left(\begin{array}{cc:c} sI-A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI-\tilde{A}_{33} & -B_{32} \\ \hdashline 0 & I & 0 \end{array}\right).$$

Since the first matrix on the left is unimodular and the second matrix has full row rank for all $s\in\mathbb{C}$ (see Lemma 4.2), the matrix on the right has full row rank for all $s\in\mathbb{C}$. Hence the system (6.9) is strongly controllable.

Now consider the almost disturbance decoupling problem for the system (6.5) with output q_3 and "disturbance" $\binom{x_1}{w}$. Because of strong controllability of (6.9) there exists a feedback law $v_2=F_1\binom{x_2}{q_3}$ such that in (6.5) we have

$$\|q_3\|_2<\tfrac{1}{2}\rho(M+M_1M+\rho M_2)^{-1}\{\|w\|_2+\|x_1\|_2\} \tag{6.10}$$

for all w and x_1 in $\mathcal{L}_2$ and such that the matrix

$$\tilde{A}:=\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix}+\begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}F_1$$

satisfies $\sigma(\tilde{A})\subset\mathbb{C}^-$. Combining (6.7), (6.8), and (6.10) gives us

$$\|z\|_2<(\gamma-\tfrac{1}{2}\rho)\|w\|_2$$

for all w in $\mathcal{L}_2$. Summarizing, we have now shown that if in our original system (5.3) we apply the state feedback law

$$\begin{aligned} v_1&=-(D_1^TD_1)^{-1}B_{11}^TP_1x_1, \\ v_2&=F_1\begin{pmatrix} x_2 \\ x_3+(C_{23}^TC_{23})^{-1}(A_{13}^TP_{11}+C_{23}^TC_{21})x_1 \end{pmatrix}, \end{aligned} \tag{6.11}$$

then for all $w\in\mathcal{L}_2^l(\mathbb{R}^+)$ we have $\|z\|_2<\gamma\|w\|_2$. Thus, the H_∞ norm of the resulting closed-loop transfer matrix is smaller than γ.

It remains to be shown that the closed-loop system is internally stable. We know that

$$\|(sI-\tilde{A}_{11})^{-1}A_{13}\|_\infty\leqq M_2, \tag{6.12}$$

$$\left\|(0\quad I)(sI-\tilde{A})^{-1}\begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix}\right\|_\infty\leqq\tfrac{1}{2}\rho(M+M_1M+\rho M_2)^{-1}\leqq\tfrac{1}{2}M_2^{-1}. \tag{6.13}$$

The closed-loop A-matrix resulting from the feedback law (6.11) is given by

$$A_{\mathrm{cl}}:=\begin{pmatrix} \tilde{A}_{11} & (0\ \ A_{13}) \\ \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} & \tilde{A} \end{pmatrix}.$$

Assume $(x^T,y^T,z^T)^T$ is an eigenvector of A_{cl} with eigenvalue λ with $\operatorname{Re}\lambda\geqq 0$. It can be seen that

$$x=(\lambda I-\tilde{A}_{11})^{-1}A_{13}z, \tag{6.14}$$

$$z=(0\quad I)(\lambda I-\tilde{A})^{-1}\begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix}x. \tag{6.15}$$

(Note that the inverses exist due to the fact that $\tilde{A}_{11}$ and $\tilde{A}$ are stable matrices.)

Combining (6.12) and (6.14) we find $\|x\| \leqq M_2\|z\|$, and combining (6.13) and (6.15) yields $\|z\|_2 \leqq \frac{1}{2}M_2^{-1}\|x\|_2$. Hence $x=z=0$. This, however, would imply that $(y^T \ 0^T)^T$ is an unstable eigenvector of $\tilde{A}$. Since $\sigma(\tilde{A}) \subset \mathbb{C}^-$, this yields a contradiction. This proves that the closed-loop system is internally stable. □

A proof of the implication (ii)⇒(i) in Theorem 2.1 is now obtained by combining Theorems 5.4 and 6.2.

Remark 6.3. In the regular case (i.e., D injective) it is quite easy to give an explicit expression for a suitable state feedback law. Indeed, if $P \geqq 0$ is a solution to the algebraic Riccati equation (5.1) such that (5.2) holds, then the feedback law $u = -(D^TD)^{-1}(B^TP + D^TC)x$ achieves internal stability and $\|G_F\|_\infty < \gamma$. In the singular case (i.e., D not injective) a state feedback law is given by $u = F_0x + v$. Here, F_0 is given by (4.1) and $v = (v_1^T, v_2^T)^T$ is given by (6.11). The matrix P_{11} is obtained by solving the quadratic matrix inequality or, equivalently, by solving the reduced order Riccati equation (5.8). The matrix F_1 is a "state feedback" for the strongly controllable auxiliary system (6.5). This state feedback achieves almost disturbance decoupling between the "disturbance" $(x_1^T, w^T)^T$ and the "output" q_3. The required accuracy of decoupling is expressed by (6.10). A conceptual algorithm to construct such F_1 can be based on the proof of [19, Thm. 3.36].

7. Discussion and conclusions. In this paper we have shown that if in the H_∞ problem with state feedback *no* assumptions are made on the direct feedthrough matrix of the control input, then the central role of the algebraic Riccati equation is taken over by a quadratic matrix inequality. We note that a similar phenomenon is known to occur in the linear quadratic regulator problem: if the weighting matrix of the control input is singular, then the optimal cost is given in terms of a (linear) matrix inequality rather than in terms of an algebraic Riccati equation (see [21]). However, while in the singular LQ problem optimal inputs in general are distributions, in the H_∞ context *also in the singular case suitable state feedback laws can be found.* It is well known that in the LQ problem a special role is played by solutions of the linear matrix inequality that *minimize* the rank of the dissipation matrix (see [4], [13]). It turns out that also in our context the relevant solutions to the quadratic matrix inequality are *rank minimizing.* Indeed, it follows from the proof of Theorem 5.4 that for *all* symmetric matrices P we have rank $F_\gamma(P) \geqq$ normrank G. Thus, (2.5) can be interpreted as saying that P minimizes the rank of $F_\gamma(P)$. On the other hand, once we know that rank $F_\gamma(P) =$ normrank G, then obviously for all $s \in \mathbb{C}$ we have

$$\operatorname{rank}\begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} \leqq n + \operatorname{normrank} G.$$

Thus, (ii) of Theorem 2.1 can, loosely speaking, be reformulated as follows. There exists a solution $P \geqq 0$ to $F_\gamma(P) \geqq 0$ that *minimizes* rank $F_\gamma(P)$ and *maximizes* rank $(L_\gamma(P, s)^T, F_\gamma(P)^T)^T$ for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+$.

As can be expected, the quadratic matrix inequality and the rank conditions (2.5) and (2.6) turn out to play an important role in the context of *singular linear quadratic differential games.* This connection is elaborated in [16].

Needless to say, several questions remain unanswered in this paper. The most obvious topic is the extension of the theory of this paper to the case of dynamic measurement feedback, i.e., the singular counterpart of the problem studied in [2], [5], and [18]. In [17] it is shown that the existence of suitable dynamic compensators require solvability of a *pair* of quadratic matrix inequalities.

Finally, in [20] the ideas of the present paper are used to tackle the *finite horizon* "H_∞" control problem by measurement feedback, i.e., the problem of finding a dynamic compensator such that the $L_2[t_0, t_1]$-induced norm (instead of the $L_2(\mathbb{R}^+)$-induced norm) of the closed-loop operator is smaller than an a priori given upper bound. In [20] conditions for the existence of such a compensator are formulated in terms of quadratic differential inequalities (the extensions of Riccati differential equations).

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